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The Stability of Motion of Satellites with Cavities Partially Filled with Liquid

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by

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Abstract

This research is concerned with the stability and time-dependent motion of a spinning satellite, simulated by a rigid body with a cavity partially filled with liquid. The work includes the problem formulation, consisting of the boundary-value problem for the liquid and moment equations for the entire system. Because of large Reynold's numbers involved, viscosity effects are negligible everywhere except for a thin boundary layer near the wetted surface. Using a boundary-layer analysis, the effect of the boundary layer is replaced by modified boundary conditions for the liquid. The solution of the differential equations for the inviscid problem has been solved in closed form. A semi-analytical numerical solution of the inviscid equations subject to the viscous boundary condition has proved unsuccessful.

CONTENTS

1. Kinematical Relations

Let XYZ be an inertial coordinate system with the origin at the mass center c of the satellite (see Fig. 1), and let the satellite be rotating with an angular velocity $\bar{\omega}$ with respect to axes XYZ. Denoting by xyz a set of body axes, and by $\bar{i}$, $\bar{j}$, $\bar{k}$ the unit vectors along these axes, the angular velocity of the satellite can be written in the form

$$\bar{\omega} = \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k} \tag{1}$$

where ω_x, ω_y, ω_z are the corresponding angular velocity components.

It will prove convenient to introduce a set of axes $\xi\eta\zeta$ parallel to axes xyz but with the origin at the center 0 of the spherical tank instead of the center c of the satellite, as shown in Fig. 1. Denoting by $\bar{R}_0$ the vector from c to 0 and by $\bar{r}$ the vector from 0 to any position in the liquid, the absolute position of any point in the fluid can be written as

$$\bar{R} = \bar{R}_0 + \bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \tag{2}$$

where

$$\bar{R}_0 = R_{0x}\bar{i} + R_{0y}\bar{j} + R_{0z}\bar{k} \tag{3}$$

and

$$\bar{r} = \xi\bar{i} + \eta\bar{j} + \zeta\bar{k} \tag{4}$$

Differentiating Eq. (2) with respect to time, and recognizing the fact that the unit vectors $\bar{i}$, $\bar{j}$, $\bar{k}$ rotate with angular velocity $\bar{\omega}$,

we obtain the absolute velocity at a point in the liquid in the form

$$\overline{v} = \frac{d\overline{R}}{dt} = \frac{d}{dt} (\overline{R}_0 + \overline{r}) = \overline{v}_0 + \overline{q} + \overline{\omega} \times \overline{r} \tag{5}$$

where $\overline{v}_0$ is the velocity of the origin 0 and

$$\overline{q} = \frac{d\xi}{dt} \overline{i} + \frac{d\eta}{dt} \overline{j} + \frac{d\zeta}{dt} \overline{k} \tag{6}$$

is the velocity of the liquid relative to the tank. The absolute acceleration of the liquid can be obtained by differentiating Eq. (5) with respect to time. The result is

$$\overline{a} = \frac{d\overline{v}}{dt} = \overline{a}_0 + \frac{D\overline{q}}{Dt} + \overline{\omega} \times q + \frac{d\overline{\omega}}{dt} \times \overline{r} + \overline{\omega} \times \frac{d\overline{r}}{dt} \tag{7}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{q} \cdot \nabla \tag{8}$$

is known as the substantial derivative operator. Denoting $d\overline{\omega}/dt$ by $\dot{\overline{\omega}}$, and inserting Eq. (8) into Eq. (7), we can express the absolute acceleration at a point in the liquid as

$$\overline{a} = \overline{a}_0 + \frac{\partial\overline{q}}{\partial t} + \overline{q} \cdot \nabla\overline{q} + 2\overline{\omega} \times \overline{q} + \dot{\overline{\omega}} \times \overline{r} + \overline{\omega} \times (\overline{\omega} \times \overline{r}) \tag{9}$$

in which $\overline{q} \cdot \nabla\overline{q}$ is known as the convective acceleration. The absolute acceleration of a point in the rigid part of the satellite can be obtained by letting $\overline{q} = \overline{0}$ in Eq. (9).

2. The Mathematical Formulation for the Motion of the System

The problem formulation consists of the boundary-value problem, namely, the differential equation and boundary condition for the liquid, and the moment equation governing the rotation of the satellite.

2.1 The equations governing the liquid motion

For an incompressible fluid, the velocity vector $\bar{q}$ must satisfy the continuity equation

$$\nabla \cdot \bar{q} = 0 \tag{10}$$

In this case, the Navier-Stokes equations become

$$\bar{a} = \bar{a}_0 + \frac{\partial \bar{q}}{\partial t} + \bar{q} \cdot \nabla \bar{q} + 2\bar{\omega} \times \bar{q} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})$$

$$= - \frac{1}{\rho} \nabla p + \bar{f} + \nu \nabla^2 \bar{q} \tag{11}$$

where $\bar{f}$ is the body force per unit mass, and ρ, p, and ν are the density, pressure, and kinematic viscosity, respectively.

From vector algebra, we have

$$\bar{q} \cdot \nabla \bar{q} = \frac{1}{2} \nabla(\bar{q} \cdot \bar{q}) - \bar{q} \times (\nabla \times \bar{q})$$

$$\bar{\omega} \times (\bar{\omega} \times \bar{r}) = - \frac{1}{2} \nabla[(\bar{\omega} \times \bar{r}) \cdot (\bar{\omega} \times \bar{r})] \tag{12}$$

$$\bar{a}_0 = \nabla(\bar{a}_0 \cdot \bar{r})$$

where $\nabla \times \bar{q}$ is known as the vorticity of the liquid relative to the tank. Assuming that the body force is derivable from a potential function, we can write

$$\bar{f} = - \nabla B \tag{13}$$

where B is a scalar function. It is convenient to introduce a pressure function p* defined by

$$p^* = \frac{1}{\rho} p + B + \bar{a}_U \quad \bar{r} + \frac{1}{2} \bar{q} \cdot \bar{q} - \frac{1}{2} (\bar{\omega} \times \bar{r}) \cdot (\bar{\omega} \times \bar{r}) \tag{14}$$

Then, Eq. (11) can be rewritten as

$$\frac{\partial \bar{q}}{\partial t} - \bar{q} \times (\nabla \times \bar{q}) + 2\bar{\omega} \times \bar{q} + \dot{\bar{\omega}} \times \bar{r} = - \nabla p^* + \nu \nabla^2 \bar{q} \tag{15}$$

2.2 Boundary conditions

The boundary conditions on the liquid are of two types, namely, geometric and natural. The geometric boundary conditions are the result of kinematic considerations, whereas the natural boundary conditions are the result of dynamic considerations. Moreover, we must distinguish between the boundary conditions on the wetted surface S_w and the free surface S_f.

At the wetted surface the velocity of the liquid relative to the satellite is zero. If $\bar{n}$, $\bar{t}_1$, and $\bar{t}_2$ denote unit vectors normal and tangential to the wetted surface, then we have

$$q_n = \overline{q} \cdot \overline{n} = 0 \qquad \text{on } S_w \tag{16a}$$

$$q_{t1} = \overline{q} \cdot \overline{t}_1 = 0, \quad q_{t2} = \overline{q} \cdot \overline{t}_2 = 0 \qquad \text{on } S_w \tag{16b}$$

The boundary condition (16a) is a result of the impermeability of the rigid part of the satellite and must be satisfied whether the fluid is viscous or inviscid. On the other hand, the boundary conditions (16b) are referred to as the no-slip conditions because, irrespective of how small the viscosity of the fluid is, there can be no relative tangential velocity between the fluid and the wetted surface. If the fluid is inviscid, the boundary conditions (16b) may not be satisfied.

At the free surface there are three boundary conditions. The first is kinematical in nature, expressing the fact that a particle on the free surface must always remain on the free surface that is, if $F(x,y,z,t) = 0$ is the equation of the free surface, the we must have

$$\frac{D}{Dt}[F(x,y,z,t)] = 0 \quad \text{on} \quad F(x,y,z,t) = 0 \tag{17}$$

Recalling definition (8), Eq. (17) becomes

$$\frac{\partial F}{\partial t} + q_x \frac{\partial F}{\partial x} + q_y \frac{\partial F}{\partial y} + q_z \frac{\partial F}{\partial z} = 0 \quad \text{on} \quad F(x,y,z,t) = 0 \tag{18}$$

where q_x, q_y, q_z are the components of $\overline{q}$ along the xyz axes. The other boundary conditions are dynamic, representing the continuity of normal and tangential stresses across the free interface. Taking the surface

tension into account, we can write the continuity of the normal stresses

$$T(K_1 + K_2) + P = \sigma \quad \text{on} \quad F(x,y,z,t) = 0 \tag{19}$$

where T is the surface tension, K_1 and K_2 are the curvatures of the free surface along two orthogonal directions, P represents the pressure in the vacant part of the tank, and σ is the normal stress. The continuity of the tangential stresses is expressed by

$$\tau_1 = 0 \quad , \quad \tau_2 = 0 \quad \text{on} \quad F(x,y,z,t) \tag{20}$$

2.3 The moment equation for the satellite

We assume that the mass center c of the satellite is fixed in an inertial space, and, moreover, that it does not shift with respect to the rigid part of the satellite, even though the liquid may be moving. Recognizing that the acceleration of any particle in the rigid part of the satellite can be obtained by letting $\overline{q} = \overline{0}$ in Eq. (9), we can write the moment equation about c in the form

$$\int_{m_\ell} \overline{R} \times [\overline{a}_0 + \frac{\partial \overline{q}}{\partial t} + \overline{q} \cdot \nabla \overline{q} + 2\overline{\omega} \times \overline{q} + \dot{\overline{\omega}} \times \overline{r} + \overline{\omega} \times (\overline{\omega} \times \overline{r})]dm_\ell +$$

$$\int_{m_r} \overline{R} \times [\overline{a}_0 + \dot{\overline{\omega}} \times \overline{r} + \overline{\omega} \times (\overline{\omega} \times \overline{r})]dm_r =$$

$$\int_{m_\ell} \overline{R} \times [\nabla(\frac{1}{\rho} p + B) + \nu \nabla^2 \overline{q}]dm_\ell + \int_{m_r} \overline{R} \times \overline{f}_r \, dm_r \tag{21}$$

where the subscripts ℓ and r designate the liquid and rigid parts of the satellite, respectively. Because the acceleration of the point 0 is

$$\bar{a}_0 = \dot{\bar{\omega}} \times \bar{R}_0 + \bar{\omega} \times (\bar{\omega} \times \bar{R}_0) \tag{22}$$

and recalling that $\bar{R}_0 + \bar{r} = \bar{R}$, we can write*

$$\int_{m_\ell} \bar{R} \times [\dot{\bar{\omega}} \times \bar{R} + \bar{\omega} \times (\bar{\omega} \times \bar{R})]dm_\ell + \int_{m_r} \bar{R} \times [\dot{\bar{\omega}} \times \bar{R} + \bar{\omega} \times (\bar{\omega} \times \bar{R})]dm_r$$

$$= \bar{I} \cdot \dot{\bar{\omega}} + \bar{\omega} \times \bar{I} \cdot \bar{\omega} \tag{23}$$

where $\bar{I}$ is the inertia dyadic of the satellite as if it were entirely rigid. Using Eqs. (12), (14), and (23), we can rewrite Eq. (21) as

$$\bar{I} \cdot \dot{\bar{\omega}} + \bar{\omega} \times \bar{I} \cdot \bar{\omega} = - \int_{m_\ell} \bar{R} \times \{\nabla[p^* - \bar{a}_0 \cdot \bar{r} + \frac{1}{2} (\bar{\omega} \times \bar{r}) \cdot (\bar{\omega} \times \bar{r})] -$$

$$\bar{q} \times (\nabla \times \bar{q}) + \frac{\partial \bar{q}}{\partial t} + 2\bar{\omega} \times \bar{q} + \nu\nabla^2 q\}dm_\ell + \int_{m_r} \bar{R} \times \bar{f}_r \, dm_r \tag{24}$$

where the first integral on the right side of Eq. (24) is the disturbing torque due to the liquid motion and the second integral is due to body forces acting on the rigid part.

3. Steady-State Equilibrium Configuration

First we seek an equilibrium configuration in which the satellite rotates with a constant angular velocity Ω about the inertial axis Z

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* See Ref. 1, Sec. 12.8

and in which the liquid is at rest with respect to the satellite. Moreover, the body force is assumed to be zero. To describe this configuration, we introduce a coordinate system x'y'z' rotating with the angular velocity Ω about the axis Z, and denote by $\bar{i}'$, $\bar{j}'$, $\bar{k}'$ the unit vectors along these axes. Hence, the equilibrium configuration is given by

$$\bar{q} = 0 \ , \quad \bar{\omega} = \Omega\bar{k}' \ , \quad \dot{\bar{\omega}} = 0 \ , \quad F_0(x,y,z) = 0 \tag{25}$$

where $F_0(x,y,z) = 0$ is the equation of the free surface.

Substituting Eq. (25) into Eqs. (14) and (15), and recalling Eq. (22), we obtain

$$p^* = \frac{1}{\rho} p' - \frac{1}{2} \Omega^2(x'^2 + y'^2) = \text{const} \tag{26}$$

The boundary conditions (16), (18), and (20) are automatically satisfied, whereas, boundary condition (19), together with Eq. (26), yields

$$T(K_1 + K_2) + \frac{1}{2} \rho \, \Omega^2(x'^2 + y'^2) = \text{const} \tag{27}$$

which describes the free surface $F_0(x', y', z') = 0$. Hence, the free surface is the intersection of the curve given by Eq. (27) and the sphere

$$(x' - R_{0x'})^2 + (y' - R_{0y'})^2 + (z' - R_{0z'})^2 = R_s^2 \tag{28}$$

where R_s is the radius of the spherical tank. In the case in which the surface tension T is negligible, the free surface is given by the inter-

section of the cylinder

$$x'^2 + y'^2 = R_c^2 \tag{28}$$

and the spherical tank given by Eq. (28), where the radius R_c of the cylinder is a function of the volume of liquid in the tank.

4. Linear Perturbation Problem

Next, let us consider the case of a small perturbation from the steady-state solution discussed in Section 3. To this end, we write

$$\overline{\omega} = \Omega\overline{k}' + \varepsilon\overline{\omega}_1 + \ldots , \quad \dot{\overline{\omega}} = \varepsilon\dot{\overline{\omega}}_1 + \ldots \tag{30}$$

$$\overline{R} = \overline{R}' + \varepsilon\, \overline{R}_1 + \ldots \tag{31}$$

$$\overline{q} = \varepsilon\overline{q}_1 + \ldots \tag{32}$$

$$p = p' + \varepsilon\, p_1 + \ldots \tag{33}$$

$$F(x,y,z,t) = R - R_c + \varepsilon\, F_1(\phi,z,t) + \ldots = 0 \tag{34}$$

where ε is a small parameter and R, ϕ, z a cylindrical coordinate system with the origin at c. Recognizing that the perturbed body is inclined slightly with respect to axes $x'y'z'$, we let the body axes triad xyz be obtained from $x'y'z'$ by means of the small rotations $\varepsilon\theta_1$ about x', $\varepsilon\theta_2$ about y', and $\varepsilon\theta_3$ about z', which enables us to write

$$\begin{aligned} x &= x' + \varepsilon\,(\theta_3 y' - \theta_2 z') + \ldots \\ y &= y' + \varepsilon\,(\theta_1 z' - \theta_3 x') + \ldots \\ z &= z' + \varepsilon\,(\theta_2 x' - \theta_2 y') + \ldots \end{aligned} \tag{35}$$

so that, from Eqs. (31) and (35), it follows that

$$\overline{R}_1 = (\theta_3 y' - \theta_2 z')\ \overline{i} + (\theta_1 z' - \theta_3 x')\ \overline{j} + (\theta_2 x' - \theta_1 y')\ \overline{k} \tag{36}$$

Substituting Eqs. (30) - (34) into Eqs. (14), (15), (19) - (20), and (24), and equating coefficients of equal powers of ε, we obtain the steady-state solution of Section 3, corresponding to the $O(\varepsilon^0)$ problem, and the following equations for the $O(\varepsilon)$ problem

$$\nabla \cdot \overline{q}_1 = 0 \tag{37}$$

$$\frac{\partial \overline{q}_1}{\partial t} + 2\Omega\overline{k} \times \overline{q}_1 + \dot{\overline{\omega}}_1 \times \overline{R}' =$$

$$-\nabla[\frac{1}{\rho} p_1 - (\Omega\overline{k}' \times \overline{R}')\cdot(\overline{\omega}_1 \times \overline{R}' + \Omega\overline{k}' \times \overline{R}_1)] + \nu \nabla^2 \overline{q}_1 \tag{38}$$

$$\overline{I} \cdot \dot{\overline{\omega}}_1 + \Omega\overline{k}' \times \overline{I} \cdot \overline{\omega}_1 + \overline{\omega}_1 \times \overline{I} \cdot \Omega\overline{k}' =$$

$$- \int_{m_\ell} [\overline{R}' \times (\frac{1}{\rho} \nabla p_1 + \frac{\partial \overline{q}_1}{\partial t} + 2\Omega\overline{k} \times \overline{q}_1 - \nu \nabla^2 \overline{q}_1) + \overline{R}_1 \times \frac{1}{\rho} \nabla p']dm_\ell$$

(39)

where $\overline{q}_1$ is subject to the following boundary conditions

$$q_r = 0 \qquad \text{on} \quad r = R_s \tag{40a}$$

$$q_\alpha = q_\beta = 0 \qquad \text{on} \quad r = R_s \tag{40b}$$

$$q_R + \frac{\partial F_1}{\partial t} = 0 \qquad \text{on} \quad R = R_c \tag{41}$$

$$\sigma_R = -p_1 + 2\rho\nu \frac{\partial q_R}{\partial R} = 0 \qquad \text{on} \quad R = R_c \tag{42a}$$

$$\tau_{Rz} = \rho\nu\left(\frac{\partial q_R}{\partial z} + \frac{\partial q_z}{\partial R}\right) = 0 \qquad \text{on} \quad R = R_c \tag{42b}$$

$$\tau_{R\phi} = \rho\nu\left[R \frac{\partial}{\partial R}\left(\frac{q_\phi}{R}\right) + \frac{1}{R}\frac{\partial q_R}{\partial \phi}\right] = 0 \qquad \text{on} \quad R = R_c \tag{42c}$$

where r, α, β is a spherical coordinate system centered at 0, whereas R, ϕ, z is a cylindrical coordinate system centered at c. Note that the boundary condition (41) was obtained by inserting Eq. (34) into the kinematic condition (18), performing a so-called "transfer of boundary conditions", and keeping the $O(\varepsilon)$ terms*.

5. Method of Solution

5.1 The equations of motion in dimensionless form

To solve Eqs. (37) - (39) subject to the boundary conditions (40) - (42), we first consider the fluid problem. To this end, we introduce the dimensionless quantities

$$\begin{aligned}
&\hat{x} = x'/2R_s \ , \quad \hat{y} = y'/2R_s \ , \quad \hat{z} = z'/2R_s \\
&\hat{t} = t\omega_N \ , \quad \hat{\Omega} = \Omega/\omega_N \\
&F_1 = R_s \hat{F}_1 \, e^{\hat{\lambda}\hat{t}} \\
&\bar{q}_1 = \hat{\bar{q}}_1 \, U \, e^{\hat{\lambda}\hat{t}} \\
&p_1 = \hat{p}_1 \, \rho U^2 \, e^{\hat{\lambda}\hat{t}}
\end{aligned} \tag{43}$$

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* See Ref. 2, Sec. 2.1.3

$$\bar{\omega}_1 = \dot{\theta}_1\ \bar{i} + \dot{\theta}_2\ \bar{j} + \dot{\theta}_3\ \bar{k} = \hat{\lambda}\omega_N\ \hat{\bar{\theta}}\ e^{\hat{\lambda}\hat{t}}$$

$$\bar{R}_1 = \hat{\bar{R}}_1\ \frac{U}{\omega_N}\ e^{\hat{\lambda}\hat{t}}$$

where $U = 2R_s\omega_N$ is a fictitious velocity and $\omega_N = \Omega(C-A)/A$ is the nutational frequency of the rigid satellite, in which A is the principal moment of inertia about x or y and C the principal moment of inertia about z. Moreover, $\hat{\lambda}$ is the dimensionless eigenvalue sought. Substituting Eqs. (43) into Eqs. (37) and (38), we obtain

$$\hat{\nabla} \cdot \bar{q}_1 = 0 \tag{44}$$

$$\hat{\lambda}\hat{\bar{q}}_1 + 2\hat{\Omega}\bar{k}' \times \hat{\bar{q}}_1 + \hat{\lambda}^2\hat{\bar{\theta}} \times \hat{\bar{R}}' =$$

$$- \hat{\nabla}[\hat{p}_1 - (\hat{\Omega}\bar{k}' \times \hat{\bar{R}}')\cdot(\hat{\lambda}\hat{\bar{\theta}} \times \hat{\bar{R}}' + \hat{\Omega}\bar{k}' \times \hat{\bar{R}}_1)] + \frac{1}{R_e}\ \hat{\nabla}^2\hat{\bar{q}}_1 \tag{45}$$

where, in terms of cartesian components,

$$\hat{\nabla} = \bar{i}\ \frac{\partial}{\partial\hat{x}} + \bar{j}\ \frac{\partial}{\partial\hat{y}} + \bar{k}\ \frac{\partial}{\partial\hat{z}}$$

represents a dimensionless nabla, and $R_e = 4R_s^2\ \omega_N/\nu$ is Reynold's number.

In a similar fashion, the moment equation, Eq. (39), becomes

$$\hat{\lambda}^2\hat{\bar{I}} \cdot \hat{\bar{\theta}} + \hat{\lambda}\hat{\Omega}(\bar{k}' \times \hat{\bar{I}} \cdot \hat{\bar{\theta}} + \hat{\bar{\theta}} \times \hat{\bar{I}} \cdot \bar{k}') =$$

$$- \int_{\hat{m}_\ell} [\hat{\bar{R}}' \times (\hat{\nabla}\hat{p}_1 + \hat{\lambda}\hat{\bar{q}}_1 + 2\hat{\Omega}\bar{k}' \times \hat{\bar{q}}_1 - \frac{1}{R_e}\ \hat{\nabla}^2\hat{\bar{q}}_1) + \hat{\bar{R}}_1 \times \hat{\nabla}\hat{p}_1]d\hat{m}_\ell \tag{46}$$

where $\hat{\bar{I}} = \bar{I}/4R_s^2 m$ is the dimensionless inertia dyadic of the complete system as if the system were entirely rigid, in which m is the total mass (rigid plus liquid) of the system. Moreover, $\hat{m}_\ell = m_\ell/m$ is the dimensionless mass of the liquid. For simplicity, $\hat{\bar{I}}$ is evaluated by regarding the mass of the liquid as frozen in the equilibrium configuration.

Typical values of R_s, ω_N and ν show that $R_e = O(10^5)$ or larger*. Hence, one can neglect the last term in Eq. (45) and obtain

$$\hat{\lambda}\hat{\bar{q}}_1 + 2\hat{\Omega}\bar{k}' \times \hat{\bar{q}}_1 + \hat{\lambda}^2\hat{\bar{\theta}} \times \hat{\bar{R}}' = -\hat{\nabla}[\hat{p}_1 - (\hat{\Omega}\bar{k} \times \hat{\bar{R}}')\cdot(\hat{\lambda}\hat{\bar{\theta}} \times \hat{\bar{R}}' + \hat{\Omega}\bar{k}' \times \hat{\bar{R}}_1)] \tag{47}$$

However, Eq. (47) is of first order in the spatial variables rather than second order. Hence, the general solutions of Eqs. (44) and (47) cannot be expected to satisfy all the boundary conditions (40) - (42). Because letting $R_e \to \infty$ is equivalent to assuming an inviscid fluid, the no-slip boundary conditions (40b) and the shear boundary conditions (42b) and (42c) cannot be satisfied in general.

To obtain a solution valid everywhere, we must supplement the solutions of Eqs. (44) and (47) by two boundary layers, one near the free surface and the other near the wetted surface, by using the method of composite expansions.**

The result of the boundary layer analysis will be the modification of the boundary conditions for the inviscid problem by accounting for

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* See Ref. 3

** See Ref. 2, Sec. 4.2

the liquid viscosity. Hence, the problem will reduce to the solution of an inviscid problem, with the effect of the viscosity reflected in the modified boundary conditions.

The nature of the geometry of the liquid boundaries demands the use of a cylindrical coordinate system with the origin at c to describe the boundary layer near the free surface and a spherical coordinate system with the origin at O to describe the boundary layer near the wetted surface. Note that either coordinate system can be used away from the boundaries. As a result, we shall seek the solution in two parts, one valid everywhere except near the free surface and the other valid everywhere except near the wetted surface.

5.2 The boundary layer next to the free surface

We assume the solution next to the free surface to have the form

$$\hat{\bar{q}}_1 = \hat{q}_R\bar{e}_R + \hat{q}_\phi\bar{e}_\phi + \hat{q}_z\bar{e}_z = [u_i(\hat{R},\phi,\hat{z}) + \delta u_v(\hat{\rho},\phi,\hat{z})]\bar{e}_R + [v_i(\hat{R},\phi,\hat{z}) + v_v(\hat{\rho},\phi,\hat{z})]\bar{e}_\phi + [w_i(\hat{R},\phi,\hat{z}) + w_v(\hat{\rho},\phi,\hat{z})]\bar{e}_z \tag{48}$$

$$\hat{p}_1 = p_i(\hat{R},\phi,\hat{z}) + p_v(\hat{\rho},\phi,\hat{z}) \tag{49}$$

where $\hat{\rho} = (\hat{R} - \hat{R}_c)/\delta$, in which $\delta = 1/\sqrt{R_e}$ is proportional to the boundary layer thickness. Note that the subscripts i and v refer to inviscid and viscous solutions. According to this method, we force the quantities with subscript v to tend to zero as $\hat{\rho} \to \infty$. Substituting Eqs. (48) and (49)

into Eqs. (44) and (4.), and letting $\delta \to 0$ while holding $\hat{R}$ fixed, we obtain Eqs. (44) and (47). Substituting Eqs. (48) and (49) into Eqs. (44) and (45), taking the limit $\delta \to 0$ with $\hat{\rho}$ fixed, subtracting the quantities pertaining to the inviscid solution, and using Eqs. (44) and (47), we obtain

$$\frac{\partial u_v}{\partial \hat{\rho}} + \frac{1}{\hat{R}_c} \frac{\partial v_v}{\partial \phi} + \frac{\partial w_v}{\partial \hat{z}} = 0 \tag{50}$$

$$\frac{\partial p_v}{\partial \hat{\rho}} = 0 \tag{51}$$

$$\frac{\partial^2 v_v}{\partial \hat{\rho}^2} - \hat{\lambda} v_v = \frac{1}{\hat{R}_c} \frac{\partial p_v}{\partial \phi} \tag{52}$$

$$\frac{\partial^2 w_v}{\partial \hat{\rho}^2} - \hat{\lambda} w_v = \frac{\partial p_v}{\partial \hat{z}} \tag{53}$$

The solution of Eq. (51) tending to zero as $\hat{\rho} \to \infty$ is $p_v \equiv 0$. It follows that the solutions of Eqs. (52) and (53) tending to zero as $\hat{\rho} \to \infty$ are

$$v_v = a_1(\phi, \hat{z})\, e^{\sqrt{\hat{\lambda}}\, \hat{\rho}} \tag{54}$$

$$w_v = a_2(\phi, \hat{z})\, e^{\sqrt{\hat{\lambda}}\, \hat{\rho}} \tag{55}$$

where the real part of $\sqrt{\hat{\lambda}}$ is negative. Inserting solutions (54) and (55) into Eq. (50), we obtain

$$\frac{\partial u_v}{\partial \hat{\rho}} = - (\frac{1}{\hat{R}_c} \frac{\partial a_1}{\partial \phi} + \frac{\partial a_2}{\partial \hat{z}}) e^{\sqrt{\hat{\lambda}} \hat{\rho}} \tag{56}$$

Hence,

$$u_v = - \frac{1}{\sqrt{\hat{\lambda}}} (\frac{1}{\hat{R}_c} \frac{\partial a_1}{\partial \phi} + \frac{\partial a_2}{\partial \hat{z}}) e^{\sqrt{\hat{\lambda}} \hat{\rho}} \tag{57}$$

Writing Eq. (42b) in terms of dimensionless variables, and substituting for $\hat{q}_R$ and $\hat{q}_z$ from Eq. (48), we obtain

$$\frac{\partial u_i}{\partial \hat{z}} + \delta \frac{\partial u_v}{\partial \hat{z}} + \frac{\partial w_i}{\partial \hat{R}} + \frac{1}{\delta} \frac{\partial w_v}{\partial \hat{\rho}} = 0 \tag{58}$$

Inserting Eqs. (55) and (57) into Eq. (58), we conclude that

$$a_2 = O(\delta) \tag{59}$$

Similarly, boundary condition (42c) leads to

$$a_1 = O(\delta) \tag{60}$$

In view of Eqs. (59) and (60), boundary conditions (41) and (42a) become

$$\hat{\lambda}\hat{F}_1 + 2u_i = O(\delta^2) \qquad \text{on} \quad \hat{R} = \hat{R}_c \tag{61}$$

$$\hat{p}_1 = O(\delta^2) \qquad \text{on} \quad \hat{R} = \hat{R}_c \tag{62}$$

Hence, to order δ, the boundary conditions at the free surface are the same as for the inviscid liquid and no dissipation takes place in the boundary layer next to the free surface.

5.3 The boundary layer next to the wetted surface

We seek a solution valid everywhere except near the free surface. To this end, we use spherical coordinates and write the solution in the form

$$\hat{\bar{q}}_1 = \hat{q}_r\bar{e}_r + \hat{q}_\alpha\bar{e}_\alpha + \hat{q}_\beta\bar{e}_\beta = [u_i(\hat{r},\alpha,\beta) + \delta u_v(\hat{\rho},\alpha,\beta)]\bar{e}_r$$

$$+ [v_i(\hat{r},\alpha,\beta) + v_v(\hat{\rho},\alpha,\beta)]\bar{e}_\alpha + [w_i(\hat{r},\alpha,\beta) + w_v(\hat{\rho},\alpha,\beta)]\bar{e}_\beta \tag{63}$$

where the viscous terms, namely, the components with subscript v, tend to zero as $\hat{\rho} \to \infty$, in which $\hat{\rho} = (1 - \hat{r})/\delta$. The inviscid components, namely, the components with subscript i, are solutions of Eq. (47).

To determine the viscous components, we substitute Eq. (63) into Eqs. (44) and (45), subtract the inviscid components, let $\delta \to 0$ with $\hat{\rho}$ fixed, and obtain

$$-\frac{\partial u_v}{\partial \hat{\rho}} + \frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}(v_v \sin\alpha) + \frac{1}{\sin\alpha}\frac{\partial w_v}{\partial\beta} = 0 \tag{64}$$

$$\frac{\partial \hat{p}_v}{\partial \hat{\rho}} = 0 \tag{65a}$$

$$\hat{\lambda} v_v - 2\Omega w_v \cos\alpha = -\frac{\partial \hat{p}_v}{\partial\alpha} + \frac{\partial^2 v_v}{\partial \hat{\rho}^2} \tag{65b}$$

$$\hat{\lambda} w_v + 2\Omega v_v \cos \alpha = - \frac{1}{\sin \alpha} \frac{\partial \hat{p}_v}{\partial \beta} + \frac{\partial^2 w_v}{\partial \hat{\rho}^2} \tag{65c}$$

The solution of Eq. (65a) that tends to zero as $\hat{\rho} \to \infty$ is

$$\hat{p}_v = 0 \tag{66}$$

from which it follows that the solution of Eqs. (65b) and (65c) is

$$v_v = c_1(\alpha,\beta) e^{s_1 \hat{\rho}} + c_2(\alpha,\beta) e^{s_2 \hat{\rho}} \tag{67a}$$

$$w_v = i[c_1(\alpha,\beta) e^{s_1 \hat{\rho}} - c_2(\alpha,\beta) e^{s_2 \hat{\rho}}] \tag{67b}$$

where

$$\left.\begin{matrix} s_1^2 \\ \\ s_2^2 \end{matrix}\right\} = \hat{\lambda} \mp i 2\Omega \cos \alpha \tag{68}$$

in which $i = \sqrt{-1}$. Note that the real parts of s_1 and s_2 must be negative for v_v and w_v to tend to zero as $\hat{\rho} \to \infty$. Introducing Eqs.(67) into Eq. (64), and solving for u_v, we obtain

$$u_v = \frac{1}{s_1} [- \frac{c_1}{s_1} \frac{\partial s_1}{\partial \alpha} + \frac{1}{\sin \alpha} \frac{\partial}{\partial \alpha} (c_1 \sin \alpha) + \frac{i}{\sin \alpha} \frac{\partial c_1}{\partial \beta}] e^{s_1 \hat{\rho}} +$$

$$\frac{c_1}{s_1} \frac{\partial s_1}{\partial \alpha} \hat{\rho}\, e^{s_1 \hat{\rho}} + \frac{1}{s_2} [- \frac{c_2}{s_2} \frac{\partial s_2}{\partial \alpha} + \frac{1}{\sin \alpha} \frac{\partial}{\partial \alpha} (c_2 \sin \alpha) - \frac{i}{\sin \alpha} \frac{\partial c_2}{\partial \beta}] e^{s_2 \hat{\rho}}$$

$$+ \frac{c_2}{s_2} \frac{\partial s_2}{\partial \alpha} \hat{\rho}\, e^{s_2 \hat{\rho}} = 0 \tag{69}$$

Boundary conditions (40) demand that $\hat{\vec{q}}_1 = \vec{0}$ on $\hat{r} = 1$, so that

$$u_i + \delta u_v = 0 \qquad \text{on} \quad \hat{r} = 1 \tag{70a}$$

$$v_i + v_v = 0 \qquad \text{on} \quad \hat{r} = 1 \tag{70b}$$

$$w_i + w_v = 0 \qquad \text{on} \quad \hat{r} = 1 \tag{70c}$$

Introducing Eqs. (67) and (69) into Eqs. (70), and recalling that $\hat{\rho} = 0$ when $\hat{r} = 1$, we have

$$u_i + \delta(f_1 + f_2) = 0 \qquad \text{on} \quad \hat{r} = 1 \tag{71a}$$

$$v_i + c_1 + c_2 = 0 \qquad \text{on} \quad \hat{r} = 1 \tag{71b}$$

$$w_i + i(c_1 - c_2) = 0 \qquad \text{on} \quad \hat{r} = 1 \tag{71c}$$

where for convenience we adopted the notation

$$f_j = \frac{1}{s_j}\left[-\frac{c_j}{s_j}\frac{\partial s_j}{\partial\alpha} + \frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}(c_j \sin\alpha) + \frac{i}{\sin\alpha}\frac{\partial c_j}{\partial\beta}\right],\ j = 1,2 \tag{72}$$

Equations (71b) and (71c) can be solved for c_1 and c_2, with the result

$$c_1 = -\frac{1}{2}(v_i - iw_i), \quad c_2 = c_1^* = -\frac{1}{2}(v_i + iw_i) \tag{73}$$

so that c_2 is the complex conjugate of c_1. Moreover, from Eq. (68), it is easy to verify that

$$\frac{\partial s_1}{\partial \alpha} = \frac{i\Omega \sin \alpha}{s_1} \quad , \quad \frac{\partial s_2}{\partial \alpha} = - \frac{i\Omega \sin \alpha}{s_2} \tag{74}$$

Inserting Eqs. (72) - (74) into Eq. (71a), we obtain

$$u_i + \frac{\delta}{2} i\Omega \sin \alpha \, [(\frac{1}{s_1^3} - \frac{1}{s_2^3})v_i - i(\frac{1}{s_1^3} + \frac{1}{s_2^3})w_i] - \frac{\delta}{2} \cot \alpha \times$$

$$[(\frac{1}{s_1} + \frac{1}{s_2})v_i - i(\frac{1}{s_1} - \frac{1}{s_2})w_i] - \frac{\delta}{2} [(\frac{1}{s_1} + \frac{1}{s_2})\frac{\partial v_i}{\partial \alpha} - i(\frac{1}{s_1} - \frac{1}{s_2})\frac{\partial w_i}{\partial \alpha}]$$

$$- \frac{\delta}{2} \frac{i}{\sin \alpha} [(\frac{1}{s_1} - \frac{1}{s_2}) \frac{\partial v_i}{\partial \beta} - i(\frac{1}{s_1} + \frac{1}{s_2}) \frac{\partial w_i}{\partial \beta}] = 0 \quad \text{on} \quad \hat{r} = 1 \tag{75}$$

It will prove convenient to introduce the notation

$$s_1 = s \, e^{-i\gamma} \, , \quad s_2 = s \, e^{i\gamma} \tag{76}$$

where

$$s = (\lambda^2 + 4\Omega^2 \cos^2 \alpha)^{1/2}, \; \gamma = \frac{1}{2} \tan^{-1} \frac{2\Omega \cos \alpha}{\lambda} \tag{77}$$

Then, the boundary condition at the wetted surface reduces to

$$u_i - \delta \frac{\Omega \sin \alpha}{s^3} (v_i \sin 3\gamma - w_i \cos 3\gamma) - \delta \frac{\cot \alpha}{s} (v_i \cos \gamma - w_i \sin \gamma)$$

$$- \delta \frac{1}{s} (\frac{\partial v_i}{\partial \alpha} \cos \gamma + \frac{\partial w_i}{\partial \alpha} \sin \gamma) + \delta \frac{1}{s \sin \alpha}(\frac{\partial v_i}{\partial \beta} \sin \gamma - \frac{\partial w_i}{\partial \beta} \cos \gamma) = 0$$

$$\text{on} \quad \hat{r} = 1 \tag{78}$$

6. The Eigenvalue Problem

6.1 The solution of the equations of motion

Equations (44) and (47) must be solved simultaneously for the components of the velocity vector $\hat{\bar{q}}_1$ and pressure $\hat{p}_1$ in terms of the components of the rotation vector $\hat{\bar{\theta}}$. An examination of Eqs. (44) and (47), however, reveals that they are not in a form that permits a convenient solution, because Eq. (47) is a vector equation involving both $\hat{\bar{q}}_1$ and $\hat{p}_1$, whereas Eq. (44) is a scalar equation involving only $\hat{\bar{q}}_1$. If Eqs. (44) and (47) can be transformed into a vector equation involving only $\hat{\bar{q}}_1$ and a scalar equation involving $\hat{\bar{q}}_1$ and $\hat{p}_1$, then the vector equation can be solved independently for $\hat{\bar{q}}_1$, thereupon the scalar equation yields $\hat{p}_1$. Indeed, taking the curl and divergence of Eq. (47), and considering Eq. (44), we obtain the vector equation

$$\hat{\lambda}\hat{\nabla} \times \hat{\bar{q}}_1 - 2\hat{\Omega} \frac{\partial \hat{\bar{q}}_1}{\partial \hat{z}} + 2\hat{\lambda}^2 \hat{\bar{\theta}} = \bar{0} \tag{79}$$

and the scalar equation

$$\hat{\nabla}^2 \hat{p}_1 = \frac{2\hat{\Omega}}{\hat{\lambda}} \bar{k}' \cdot \left(2\hat{\Omega} \frac{\partial \hat{\bar{q}}_1}{\partial \hat{z}} - \hat{\lambda}^2 \hat{\bar{\theta}}\right) \tag{80}$$

respectively.

Due to the nature of the problem, the use of cylindrical coordinates is indicated in the solution of Eq. (79). Inserting $\hat{\bar{q}}_1 = \hat{q}_R \bar{e}_R + \hat{q}_\phi \bar{e}_\phi + \hat{q}_z \bar{e}_z$ and $\hat{\bar{\theta}} = \hat{\theta}_R \bar{e}_R + \hat{\theta}_\phi \bar{e}_\phi + \hat{\theta}_z \bar{e}_z$ into Eq. (79), where

$$\hat{\theta}_R = \hat{\theta}_1 \cos\phi + \hat{\theta}_2 \sin\phi$$

$$\hat{\theta}_\phi = - \hat{\theta}_1 \sin\phi + \hat{\theta}_2 \cos\phi \tag{81}$$

$$\hat{\theta}_z = \hat{\theta}_3$$

and equating the coefficients of $\overline{e}_R$, $\overline{e}_\phi$, and $\overline{e}_z$ to zero, we obtain the three scalar equations

$$\hat{\lambda}\left(\frac{1}{\hat{R}}\frac{\partial \hat{q}_z}{\partial \phi} - \frac{\partial \hat{q}_\phi}{\partial \hat{z}}\right) - 2\hat{\Omega}\frac{\partial \hat{q}_R}{\partial \hat{z}} + 2\hat{\lambda}^2\hat{\theta}_R = 0 \tag{82a}$$

$$\hat{\lambda}\left(\frac{\partial \hat{q}_R}{\partial \hat{z}} - \frac{\partial \hat{q}_z}{\partial \hat{R}}\right) - 2\hat{\Omega}\frac{\partial \hat{q}_\phi}{\partial \hat{z}} + 2\hat{\lambda}^2\hat{\theta}_\phi = 0 \tag{82b}$$

$$\hat{\lambda}\left(\frac{\partial \hat{q}_\phi}{\partial \hat{R}} + \frac{\hat{q}_\phi}{\hat{R}} - \frac{1}{\hat{R}}\frac{\partial \hat{q}_R}{\partial \phi}\right) - 2\hat{\Omega}\frac{\partial \hat{q}_z}{\partial \hat{z}} + 2\hat{\lambda}^2\hat{\theta}_z = 0 \tag{82c}$$

Equations (82a) and (82b) can be solved simultaneously for $\partial\hat{q}_R/\partial\hat{z}$ and $\partial\hat{q}_\phi/\partial\hat{z}$, with the result

$$\frac{\partial \hat{q}_R}{\partial \hat{z}} = \frac{\hat{\lambda}}{4\hat{\Omega}^2 + \hat{\lambda}^2}\left[2\hat{\Omega}\frac{1}{\hat{R}}\frac{\partial \hat{q}_z}{\partial \phi} + \hat{\lambda}\frac{\partial \hat{q}_z}{\partial \hat{R}} + 2\hat{\lambda}(2\hat{\Omega}\hat{\theta}_R - \hat{\lambda}\hat{\theta}_\phi)\right] \tag{83a}$$

$$\frac{\partial \hat{q}_\phi}{\partial \hat{z}} = \frac{\hat{\lambda}}{4\hat{\Omega}^2 + \hat{\lambda}^2}\left[\hat{\lambda}\frac{1}{\hat{R}}\frac{\partial \hat{q}_z}{\partial \phi} - 2\hat{\Omega}\frac{\partial \hat{q}_z}{\partial \hat{R}} + 2\hat{\lambda}(\hat{\lambda}\hat{\theta}_R + 2\hat{\Omega}\hat{\theta}_\phi)\right] \tag{83b}$$

Inserting Eqs. (83) into Eq. (82c) differentiated with respect to $\hat{z}$, and recognizing that $\partial\hat{\theta}_R/\partial\phi = \hat{\theta}_\phi$, $\partial\hat{\theta}_\phi/\partial\phi = - \hat{\theta}_R$, we obtain

$$\frac{\hat{\lambda}^2}{4\hat{\Omega}^2 + \hat{\lambda}^2} \left(\frac{\partial^2 \hat{q}_z}{\partial \hat{R}^2} + \frac{1}{\hat{R}} \frac{\partial \hat{q}_z}{\partial \hat{R}} + \frac{1}{\hat{R}^2} \frac{\partial^2 \hat{q}_z}{\partial \phi^2}\right) + \frac{\partial^2 \hat{q}_z}{\partial \hat{z}^2} = 0 \tag{84}$$

which is a partial differential equation for the velocity component $\hat{q}_z$ alone. Its solution can be obtained by the method of separation of variables. To this end, let

$$\hat{q}_z(\hat{R},\phi,\hat{z}) = f_R(\hat{R}) f_\phi(\phi) f_z(\hat{z}) \tag{85}$$

so that, inserting Eq.(85) into (84), and dividing through by $\hat{q}_z$, we obtain

$$\frac{\hat{\lambda}^2}{4\hat{\Omega}^2 + \hat{\lambda}^2} \left[\frac{1}{f_R} \left(\frac{d^2 f_R}{d\hat{R}^2} + \frac{1}{\hat{R}} \frac{df_R}{d\hat{R}}\right) + \frac{1}{\hat{R}^2} \frac{1}{f_\phi} \frac{d^2 f_\phi}{d\phi^2}\right] + \frac{1}{f_z} \frac{d^2 f_z}{d\hat{z}^2} = 0 \tag{86}$$

Next let

$$\frac{4\hat{\Omega}^2 + \hat{\lambda}^2}{\hat{\lambda}^2} \frac{1}{f_z} \frac{d^2 f_z}{d\hat{z}^2} = k^2 \tag{87}$$

where k is a number, and denote

$$\frac{\hat{\lambda}^2}{4\hat{\Omega}^2 + \hat{\lambda}^2} k^2 = j^2$$

so that Eq. (87) can be written in the form

$$\frac{d^2 f_z}{d\hat{z}^2} - j^2 f_z = 0 \tag{88}$$

having the solution

$$f_z = A \cosh j\hat{z} + B \sinh j\hat{z} \tag{89}$$

In view of Eq. (87), Eq. (86) reduces to

$$\frac{\hat{R}^2}{f_R}\left(\frac{d^2 f_R}{d\hat{R}^2} + \frac{1}{\hat{R}}\frac{df_R}{d\hat{R}}\right) + k^2\hat{R}^2 + \frac{1}{f_\phi}\frac{d^2 f_\phi}{d\phi^2} = 0 \tag{90}$$

leading to the two equations

$$\frac{d^2 f_\phi}{d\phi^2} + \ell^2 f_\phi = 0 \tag{91}$$

and

$$\frac{d^2 f_R}{d\hat{R}^2} + \frac{1}{\hat{R}}\frac{df_R}{d\hat{R}} + \left(k^2 - \frac{\ell^2}{\hat{R}^2}\right) f_R = 0 \tag{92}$$

where ℓ is a number. The solution of Eq. (91) is simply

$$f_\phi = C \cos \ell\phi + D \sin \ell\phi \tag{93}$$

On the other hand, Eq. (92) is recognized as a Bessel equation having the solution

$$f_R = E\, J_\ell(k\hat{R}) + F\, Y_\ell(k\hat{R}) \tag{94}$$

where J_ℓ and Y_ℓ are Bessel functions of order ℓ of the first and second kind, respectively. Combining Eqs. (89), (93), and (94), we can write

$$\hat{q}_z = (A \cosh j\hat{z} + B \sinh j\hat{z})(C \cos \ell\phi + D \sin \ell\phi)[E\, J_\ell(k\hat{R}) + F\, Y_\ell(k\hat{R})] \tag{95}$$

which represents only the homogeneous part of the solution. Before obtaining the particular solution, let us derive the homogeneous parts of $\hat{q}_R$ and $\hat{q}_\phi$. Introducing Eq. (95) into Eqs. (83), ignoring the terms in $\hat{\theta}_R$ and $\hat{\theta}_\phi$, and integrating with respect to $\hat{z}$, we obtain

$$\hat{q}_R = \frac{\hat{\lambda}}{4\hat{\Omega}^2 + \hat{\lambda}^2} \frac{1}{j} (A \sinh j\hat{z} + B \cosh j\hat{z}) \{\frac{2\hat{\Omega}\ell}{\hat{R}} (-C \sin \ell\phi + D \cos \ell\phi) \times$$

$$[E\, J_\ell(k\hat{R}) + F\, Y_\ell(k\hat{R})] + \hat{\lambda}k(C \cos \ell\phi + D \sin \ell\phi)[E\, J'_\ell(k\hat{R}) + F\, Y'_\ell(k\hat{R})]\} \tag{96}$$

and

$$\hat{q}_\phi = \frac{\hat{\lambda}}{4\hat{\Omega}^2 + \hat{\lambda}^2} \frac{1}{j} (A \sinh j\hat{z} + B \cosh j\hat{z})\{ \frac{\hat{\lambda}\ell}{\hat{R}} (-C \sin \ell\phi + D \cos \ell\phi) \times$$

$$[E\, J_\ell(k\hat{R}) + F\, Y_\ell(k\hat{R})] - 2\hat{\Omega}k(C \cos \ell\phi + D \sin \ell\phi)[E\, J'_\ell(k\hat{R}) + F\, Y'_\ell(k\hat{R})]\} \tag{97}$$

where J'_ℓ and Y'_ℓ designate derivatives of J_ℓ and Y_ℓ with respect to $k\hat{R}$.

Next let us turn our attention to the particular solution, and assume the solution in the form

$$\hat{q}_R = c_1(\phi)\hat{z} + c_2(\phi)\hat{R}, \quad \hat{q} = c_3(\phi)\hat{z} + c_4(\phi)\hat{R}, \quad \hat{q}_z = c_5(\phi)\hat{z} + c_6(\phi)\hat{R} \tag{98}$$

Introducing solution (98) into Eqs. (82), as well as into the continuity equation, Eq. (44), we obtain the set of equations

$$\hat{\lambda}[\frac{1}{R}(\frac{dc_5}{d\phi}\hat{z} + \frac{dc_6}{d\phi}\hat{R}) - c_3] - 2\hat{\Omega}c_1 + 2\hat{\lambda}^2\hat{\theta}_R = 0$$

$$\hat{\lambda}(c_1 - c_6) - 2\hat{\Omega}c_3 + 2\hat{\lambda}^2\hat{\theta}_\phi = 0$$

$$\hat{\lambda}(c_4 + c_3\frac{\hat{z}}{\hat{R}} + c_4 - \frac{1}{\hat{R}}\frac{dc_1}{d\phi}\hat{z} - \frac{dc_2}{d\phi}) - 2\hat{\Omega}c_5 + 2\hat{\lambda}^2\hat{\theta}_z = 0 \tag{99}$$

$$c_2 + c_1\frac{\hat{z}}{\hat{R}} + c_2 + \frac{1}{\hat{R}}(\frac{dc_3}{d\phi}\hat{z} + \frac{dc_4}{d\phi}\hat{R}) + c_5 = 0$$

It can be verified that the solution of Eqs. (99) can be written as

$$c_1 = -\hat{\lambda}\hat{\theta}_\phi, \quad c_2 = 0, \quad c_3 = \hat{\lambda}\hat{\theta}_R$$
$$c_4 = -\hat{\lambda}\hat{\theta}_z, \quad c_5 = 0, \quad c_6 = \hat{\lambda}\hat{\theta}_\phi - 2\hat{\Omega}\hat{\theta}_R \tag{100}$$

Hence, the complete solutions for $\hat{q}_R$, $\hat{q}_\phi$, and $\hat{q}_z$ are

$$\hat{q}_R = \frac{\hat{\lambda}}{4\hat{\Omega}^2 + \hat{\lambda}^2}\frac{1}{j}(A \sinh j\hat{z} + B \cosh j\hat{z})\{\frac{2\hat{\Omega}\ell}{\hat{R}}(-C \sin \ell\phi + D \cos \ell\phi) \times$$
$$[E\, J_\ell(k\hat{R}) + F\, Y_\ell(k\hat{R})] + \hat{\lambda}k(C \cos \ell\phi + D \sin \ell\phi)[E\, J_\ell'(k\hat{R}) + F\, Y_\ell'(k\hat{R})]\}$$
$$- \hat{\lambda}\hat{\theta}_\phi\hat{z} \tag{101}$$

$$\hat{q}_\phi = \frac{\hat{\lambda}}{4\hat{\Omega}^2 + \hat{\lambda}^2}\frac{1}{j}(A \sinh j\hat{z} + B \cosh j\hat{z})\{\frac{\hat{\lambda}\ell}{\hat{R}}(-C \sin \ell\phi + D \cos \ell\phi) \times$$
$$[E\, J_\ell(k\hat{R}) + F\, Y_\ell(k\hat{R})] - 2\hat{\Omega}k(C \cos \ell\phi + D \sin \ell\phi)[E\, J_\ell'(k\hat{R}) + F\, Y_\ell'(k\hat{R})]\}$$
$$+ \hat{\lambda}(\hat{\theta}_R\hat{z} - \hat{\theta}_z\hat{R}) \tag{102}$$

$$\hat{q}_z = (A \cosh j\hat{z} + B \sinh j\hat{z})(C \cos \ell\phi + D \sin \ell\phi) \times$$

$$[E\ J_\ell(k\hat{R}) + F\ Y_\ell(k\hat{R})] + (\hat{\lambda}\hat{\theta}_\phi - 2\hat{\Omega}\hat{\theta}_R)\hat{R} \tag{103}$$

It remains to obtain the solution for the pressure $\hat{p}_1$. Introducing solution (103) into Eq. (80), and recognizing that $\overline{k}' = \overline{e}_z$, we obtain the equation

$$\hat{\nabla}^2\hat{p}_1 = \frac{4\Omega^2}{\hat{\lambda}} j\ (A \sinh j\hat{z} + B \cosh j\hat{z})\ (C \cos \ell\phi + D \sin \ell\phi) \times$$

$$[E\ J_\ell(k\hat{R}) + F\ Y_\ell(k\hat{R})] - 2\hat{\Omega}\hat{\lambda}\hat{\theta}_z \tag{104}$$

The solution of Eq. (104) can be written in three parts, namely,

$$\hat{p}_1 = \hat{p}_2(\hat{R},\phi,\hat{z}) + \hat{p}_3(\hat{R},\phi,\hat{z}) + \hat{p}_4(\hat{R}) \tag{105}$$

where $\hat{p}_2$ satisfies the equation

$$\hat{\nabla}^2\hat{p}_2 = \frac{4\hat{\Omega}^2}{\hat{\lambda}} j\ (A \sinh j\hat{z} + B \cosh j\hat{z})(C \cos \ell\phi + D \sin \ell\phi) \times$$

$$[E\ J_\ell(k\hat{R}) + F\ Y_\ell(k\hat{R})] \tag{106}$$

$\hat{p}_3$ is the solution of Laplace's equation, and $\hat{p}_4$ satisfies

$$\hat{\nabla}^2\hat{p}_4 = \frac{d^2\hat{p}_4}{d\hat{R}^2} + \frac{1}{\hat{R}}\frac{d\hat{p}_4}{d\hat{R}} = -\ 2\hat{\Omega}\hat{\lambda}\hat{\theta}_z \tag{107}$$

Writing the solution of Eq. (106) in the form

$$\hat{p}_2 = (A \sinh j\hat{z} + B \cosh j\hat{z})(C \cos \ell\phi + D \sin \ell\phi)[E^*\ J_\ell(k\hat{R}) + F^*\ Y_\ell(k\hat{R})] \tag{108}$$

and recognizing that

$$\frac{\partial^2 \hat{p}_2}{\partial \phi^2} = - \ell^2 \hat{p}_2 \ , \quad \frac{\partial^2 \hat{p}_2}{\partial \hat{z}^2} = j^2 \hat{p}_2$$

Eq. (106) reduces to

$$[\frac{d^2}{d\hat{R}^2} + \frac{1}{\hat{R}} \frac{d}{d\hat{R}} + (j^2 - \frac{\ell^2}{\hat{R}^2})] \ [E^* \ J_\ell(k\hat{R}) + F^* \ Y_\ell(k\hat{R})]$$

$$= \frac{4\hat{\Omega}^2}{\hat{\lambda}} \ j \ [E \ J_\ell(k\hat{R}) + F \ Y_\ell(k\hat{R})] \tag{109}$$

But $E^* \ J_\ell(k\hat{R}) + F^* \ Y_\ell(k\hat{R})$ is a solution of Bessel's equation, Eq. (92), from which it follows that

$$[\frac{d^2}{d\hat{R}^2} + \frac{1}{\hat{R}} \frac{d}{d\hat{R}} + (j^2 - \frac{\ell^2}{\hat{R}^2})] \ [E^* \ J_\ell(k\hat{R}) + F^* \ Y_\ell(k\hat{R})]$$

$$= (j^2 - k^2)[E^* \ J_\ell(k\hat{R}) + F^* \ Y_\ell(k\hat{R})] = \frac{4\hat{\Omega}}{\hat{\lambda}} \ j[E \ J_\ell(k\hat{R}) + F \ Y_\ell(k\hat{R})] \tag{110}$$

Recalling the relation between j^2 and k^2, we conclude from Eq. (110) that

$$E^* = \frac{4\hat{\Omega}^2}{\hat{\lambda}} \frac{j}{j^2 - k^2} E = - \frac{\hat{\lambda}}{j} E, \quad F^* = - \frac{\hat{\lambda}}{j} F$$

Hence, solution (108) becomes

$$\hat{p}_2 = - \frac{\hat{\lambda}}{j} (A \sinh j\hat{z} + B \cosh j\hat{z})(C \cos \ell\phi + D \sin \ell\phi)[E \ J_\ell(k\hat{R}) + F \ Y_\ell(k\hat{R})] \tag{111}$$

The solution of Laplace's equation can be obtained from the solution of Eq. (86) by letting k = j. It follows immediately that

$$\hat{p}_3 = (A_1 \cosh j\hat{z} + B_1 \sinh j\hat{z})\,(C_1 \cos \ell\phi + D_1 \sin \ell\phi) \times$$
$$[E\, J_\ell(j\hat{R}) + F\, Y_\ell(j\hat{R})] \tag{112}$$

On the other hand, the solution of Eq. (107) can be written as

$$\hat{p}_4 = -\frac{1}{2}\hat{\Omega}\hat{\lambda}\hat{\theta}_z\hat{R}^2 + G \tag{113}$$

where the meaning of the constant G will become evident later. Combining Eqs. (111), (112), and (113), we obtain the pressure

$$\hat{p}_1 = -\frac{\hat{\lambda}}{j}\,(A \sinh j\hat{z} + B \cosh j\hat{z})\,(C \cos \ell\phi + D \sin \ell\phi) \times$$
$$[E\, J_\ell(k\hat{R}) + F\, Y_\ell(k\hat{R})] + (A_1 \cosh j\hat{z} + B_1 \sinh j\hat{z}) \times$$
$$(C_1 \cos \ell\phi + D_1 \sin \ell\phi)\,[E_1\, J_\ell(j\hat{R}) + F_1\, Y_\ell(j\hat{R})] - \frac{1}{2}\hat{\Omega}\hat{\lambda}\hat{\theta}_z\hat{R}^2 + G \tag{114}$$

Upon introducing solutions (101) -(103) and (114) into the moment equation, Eq. (46), and performing the appropriate integrations, it is possible to eliminate the constants $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$ from these solutions, where we note that $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$ are related to $\hat{\theta}_R$, $\hat{\theta}_\phi$, and $\hat{\theta}_z$ by Eqs. (81). To this end, it is more convenient to work with the body axes x, y, z. The inertia dyadic $\hat{\hat{I}}$ has the principal diagonal elements $\hat{I}_{11}$, $\hat{I}_{22}$, $\hat{I}_{33}$ and the off-diagonal elements $-I_{12}$, $-I_{13}$, $-I_{23}$,

$- I_{21} = - I_{12}$, $- I_{31} = - I_{13}$, $- I_{32} = - I_{23}$. Choosing the body axes to coincide with the principal axes, the products of inertia reduce to zero. Consistent with previous assumptions, we shall ignore the term R_e in Eq. (46). In view of the solutions for $\hat{\bar{q}}_1$ and $\hat{p}_1$, Eq. (46) can be written in the form of three simultaneous algebraic equations in $\hat{\theta}_1$, $\hat{\theta}_2$, and $\hat{\theta}_3$. These equations can be conveniently displayed by introducing the notation

$$\hat{M}_{11} = \int_{\hat{m}_\ell} [\hat{\lambda}^2\hat{z}^2(\cos^2\phi - \sin^2\phi) + (\hat{\Omega}^2 - \hat{\lambda}^2)\hat{R}^2 \sin^2\phi - 2\hat{\Omega}\hat{\lambda}(\hat{R}^2 - 2\hat{z}^2) \times \sin\phi \cos\phi] d\hat{m}_\ell$$

$$\hat{M}_{12} = \int_{\hat{m}_\ell} [(\hat{\Omega}^2\hat{R}^2 - \hat{\lambda}^2\hat{R}^2 - 2\hat{\lambda}^2\hat{z}^2) \sin\phi \cos\phi + 2\hat{\Omega}\hat{\lambda}\hat{R}^2 \sin^2\phi - 2\hat{\Omega}\hat{\lambda}\hat{z}^2] d\hat{m}_\ell$$

$$\hat{M}_{13} = \int_{\hat{m}_\ell} \hat{\lambda}\hat{R}\hat{z} (\hat{\lambda} \cos\phi - \hat{\Omega} \sin\phi) d\hat{m}_\ell$$

$$\hat{M}_{21} = \int_{\hat{m}_\ell} [2\hat{\Omega}\hat{\lambda}\hat{R}^2 \cos^2\phi - 2\hat{\Omega}\hat{\lambda}\hat{z}^2 - (\hat{\Omega}^2 - \hat{\lambda}^2)\hat{R}^2 \sin\phi \cos\phi] d\hat{m}_\ell \tag{115}$$

$$\hat{M}_{22} = \int_{\hat{m}_\ell} [- \hat{\lambda}^2\hat{z}^2 + (\hat{\Omega}^2 - \hat{\lambda}^2)\hat{R}^2 \cos^2\phi + 2\hat{\Omega}\hat{\lambda}\hat{R}^2 \sin\phi \cos\phi] d\hat{m}_\ell$$

$$\hat{M}_{23} = \int_{\hat{m}_\ell} \hat{\lambda}(\hat{\lambda} \sin\phi + \hat{\Omega} \cos\phi)\hat{R}\hat{z}\, d\hat{m}_\ell$$

$$\hat{M}_{31} = \int_{\hat{m}_\ell} [- (\hat{\Omega}^2 - \hat{\lambda}^2)\cos\phi + 2\hat{\Omega}\hat{\lambda} \sin\phi]\hat{R}\hat{z}\, d\hat{m}_\ell$$

$$\hat{M}_{32} = \int_{\hat{m}_\ell} [-(\hat{\Omega}^2 - \hat{\lambda}^2)\sin\phi - 2\hat{\Omega}\hat{\lambda}\cos\phi]\hat{R}\hat{z}\, d\hat{m}_\ell$$

$$\hat{M}_{33} = \int_{\hat{m}_\ell} (\hat{\Omega}^2 - \hat{\lambda}^2)\,\hat{R}^2\, d\hat{m}_\ell$$

where $\hat{M}_{ij}$ (i,j = 1, 2, 3) can be identified as terms involving the nonhomogeneous parts of $\hat{\bar{q}}_1$ and $\hat{p}_1$, and

$$\hat{H}_1 = \int_{\hat{m}_\ell} \{(\frac{\hat{z}}{\hat{R}}\frac{\partial\hat{p}_1}{\partial\phi} + \hat{\lambda}\hat{z}\hat{q}_\phi + 2\hat{\Omega}\hat{z}\hat{q}_R)\cos\phi + [\hat{z}\frac{\partial\hat{p}_1}{\partial\hat{R}} - \hat{R}\frac{\partial\hat{p}_1}{\partial\hat{z}} + \hat{\lambda}(\hat{z}\hat{q}_R - \hat{R}\hat{q}_z)$$
$$- 2\hat{\Omega}\hat{z}\hat{q}_\phi]\sin\phi\}_H\, d\hat{m}_\ell$$

$$\hat{H}_2 = \int_{\hat{m}_\ell} \{(\frac{\hat{z}}{\hat{R}}\frac{\partial\hat{p}_1}{\partial\phi} + \hat{\lambda}\hat{z}\hat{q}_\phi + 2\hat{\Omega}\hat{z}\hat{q}_R)\sin\phi - [\hat{z}\frac{\partial\hat{p}_1}{\partial\hat{R}} - \hat{R}\frac{\partial\hat{p}_1}{\partial\hat{z}} + \hat{\lambda}(\hat{z}\hat{q}_R - \hat{R}\hat{q}_z)$$
$$- 2\hat{\Omega}\hat{z}\hat{q}_\phi]\cos\phi\}_H\, d\hat{m}_\ell \qquad (116)$$

$$\hat{H}_3 = -\int_{\hat{m}_\ell} (\frac{\partial\hat{p}_1}{\partial\phi} + \hat{\lambda}\hat{R}\hat{q}_\phi + 2\hat{\Omega}\hat{R}\hat{q}_R)_H\, d\hat{m}_\ell$$

represent terms involving the homogeneous parts of $\hat{\bar{q}}_1$ and $\hat{p}_1$, where the subscript H indicates that only the homogeneous parts of the solutions for $\hat{\bar{q}}_1$ and $\hat{p}_1$ are to be included in the integrands. Considering the above, Eq. (46) can be written in the form of the three scalar equations

$$
\begin{aligned}
&(\hat{\lambda}^2\hat{I}_{11} + \hat{\lambda}\hat{\Omega}\hat{I}_{12} + \hat{M}_{11})\hat{\theta}_1 + [- \hat{\lambda}^2\hat{I}_{12} + \hat{\lambda}\hat{\Omega}(\hat{I}_{33} - \hat{I}_{22}) + \hat{M}_{12}]\hat{\theta}_2 \\
&\qquad + (- \hat{\lambda}^2\hat{I}_{13} + 2\hat{\lambda}\hat{\Omega}\hat{I}_{23} + \hat{M}_{13})\hat{\theta}_3 = \hat{H}_1 \\
&[- \hat{\lambda}^2\hat{I}_{12} + \hat{\lambda}\hat{\Omega}(\hat{I}_{11} - \hat{I}_{33}) + \hat{M}_{21}]\hat{\theta}_1 + (\hat{\lambda}^2\hat{I}_{22} - \hat{\lambda}\hat{\Omega}\hat{I}_{12} + \hat{M}_{22})\hat{\theta}_2 \\
&\qquad + (- \hat{\lambda}^2\hat{I}_{23} - 2\hat{\lambda}\hat{\Omega}\hat{I}_{13} + \hat{M}_{23})\hat{\theta}_3 = \hat{H}_2 \\
&(- \hat{\lambda}^2\hat{I}_{13} - \hat{\lambda}\hat{\Omega}\hat{I}_{23} + \hat{M}_{31})\hat{\theta}_1 + (- \hat{\lambda}^2\hat{I}_{23} + \hat{\lambda}\hat{\Omega}\hat{I}_{13} + \hat{M}_{32})\hat{\theta}_2 \\
&\qquad + (\hat{\lambda}^2\hat{I}_{33} + \hat{M}_{33})\hat{\theta}_3 = \hat{H}_3
\end{aligned}
\tag{117}
$$

Equations (117) can be solved for $\hat{\theta}_i$ $(i=1,2,3)$ to eliminate these angles from $\hat{\bar{q}}_1$ and $\hat{p}_1$.

6.2 Satisfaction of the boundary conditions

The solutions $\hat{\bar{q}}_1$ and $\hat{p}_1$ contain 13 constants of integration in addition to the numbers j and ℓ and the eigenvalue $\hat{\lambda}$. To determine these quantities, we must invoke the boundary conditions. The kinematic boundary condition, Eq. (61), merely yields the function $\hat{F}_1$ provided the solution is known. On the other hand, boundary conditions (62) and (78) can be used to determine the quantities in question.

At the free surface boundary condition (62) reduces to

$$
\hat{p}_1 = 0 \qquad \text{on} \quad \hat{R} = \hat{R}_c \tag{118}
$$

From Eq. (114), we conclude that boundary condition (118) is satisfied for all ϕ and $\hat{z}$ if

$$F = -\frac{J_\ell(k\hat{R}_c)}{Y_\ell(k\hat{R}_c)} E, \quad F_1 = -\frac{J_\ell(j\hat{R}_c)}{Y_\ell(j\hat{R}_c)} E_1, \quad G = \frac{1}{2} \hat{\Omega}\hat{\lambda}\hat{\theta}_z \hat{R}_c^2 \tag{119}$$

From the structure of the solution (114), however, we observe that the constants E and E_1 can be absorbed by A and B and A_1 and B_1, respectively. To this end, and for convenience, we let arbitrarily

$$\frac{E}{Y_\ell(k\hat{R}_c)} = \frac{E_1}{Y_\ell(j\hat{R}_c)} = 1 \tag{120}$$

so that the pressure becomes

$$\hat{p}_1 = -\frac{\hat{\lambda}}{j} (A \sinh j\hat{z} + B \cosh j\hat{z}) (C \cos \ell\phi + D \sin \ell\phi) \times$$

$$[Y_\ell(k\hat{R}_c) J_\ell(k\hat{R}) - J_\ell(k\hat{R}_c) Y_\ell(k\hat{R})] + (A_1 \cosh j\hat{z} + B_1 \sinh j\hat{z}) \times$$

$$(C_1 \cos \ell\phi + D_1 \sin \ell\phi) [Y_\ell(j\hat{R}_c) J_\ell(j\hat{R}) - J_\ell(j\hat{R}_c) Y_\ell(j\hat{R})]$$

$$+ \frac{1}{2} \hat{\Omega}\hat{\lambda}\hat{\theta}_z (\hat{R}_c^2 - \hat{R}^2) \tag{121}$$

Moreover, the velocity components become

$$\hat{q}_R = \frac{\hat{\lambda}}{4\hat{\Omega}^2 + \hat{\lambda}^2} \frac{1}{j} (A \sinh j\hat{z} + B \cosh j\hat{z}) \{ \frac{2\hat{\Omega}\ell}{\hat{R}} (-C \sin \ell\phi + D \cos \ell\phi) \times$$

$$[Y_\ell(k\hat{R}_c) J_\ell(k\hat{R}) - J_\ell(k\hat{R}_c) Y_\ell(k\hat{R})] + \hat{\lambda}k(C \cos \ell\phi + D \sin \ell\phi) \times$$

$$[Y_\ell(k\hat{R}_c) J_\ell'(k\hat{R}) - J_\ell(k\hat{R}_c) Y_\ell'(k\hat{R})] - \hat{\lambda}\hat{\theta}_\phi \hat{z}\} \tag{122}$$

$$\hat{q}_\phi = \frac{\hat{\lambda}}{4\hat{\Omega}^2 + \hat{\lambda}^2} \frac{1}{j} (A \sinh j\hat{z} + B \cosh j\hat{z}) \{\frac{\hat{\lambda}\ell}{\hat{R}} (-C \sin \ell\phi + D \cos \ell\phi)\times$$

$$[Y_\ell(k\hat{R}_c)J_\ell(k\hat{R}) - J_\ell(k\hat{R}_c)Y_\ell(k\hat{R})] - 2\hat{\Omega}k(C \cos \ell\phi + D \sin \ell\phi)\times$$

$$[Y_\ell(k\hat{R}_c)J_\ell'(k\hat{R}) - J_\ell(k\hat{R}_c)Y_\ell'(k\hat{R})] + \hat{\lambda}(\hat{\theta}_R\hat{z} - \hat{\theta}_z\hat{R})\} \tag{123}$$

$$\hat{q}_z = (A \cosh j\hat{z} + B \sinh j\hat{z})(C \cos \ell\phi + D \sin \ell\phi)[Y_\ell(k\hat{R}_c)J_\ell(k\hat{R})$$

$$- J_\ell(k\hat{R}_c)Y_\ell(k\hat{R})] + (\hat{\lambda}\hat{\theta}_\phi - 2\hat{\Omega}\hat{\theta}_R)\hat{R} \tag{124}$$

The boundary condition at the wetted surface is as given by Eq. (78). Satisfaction of this condition at every point of the boundary should yield the effects of viscosity on the stability.

7. Semi-Analytical Solution

Because the solution (121) - (124) is in terms of cylindrical coordinates with the origin at the satellite center and the boundary condition (78) is in terms of spherical coordinates with the origin at the center of the spherical tank, no exact solution is possible by separating variables. In view of this fact a semi-analytical numerical soltuion has been attempted.

The semi-analytical solution consists of solving the problem in two stages. In the first stage a solution of the eigenvalue problem is obtained by regarding the liquid as being entirely inviscid, and in the second stage the inviscid solution is perturbed, so that the perturbed motion satisfies the viscous boundary condition at the wetted surface. The inviscid solution was attempted by a Galerkin procedure, whereby a solution was assumed in the form of series of admissible functions as follows:

$$\hat{q}_R = \sum_{s=1}^{n} A_s Q_{Rs}(R,\phi,z) , \quad q_\phi = \sum_{s=1}^{n} B_s Q_{\phi s}(R,\phi,z)$$

$$\hat{q}_z = \sum_{s=1}^{n} C_s Q_{zs}(R,\phi,z) , \quad \hat{p}_1 = \sum_{s=1}^{n} D_s P_s(R,\phi,z) \tag{125}$$

where Q_{Rs}, $Q_{\phi s}$, Q_{zs}, and P_s are functions similar to solutions (121)-(124), but satisfying the boundary conditions at both the free and wetted surfaces. We note that these functions no longer satisfy the continuity equation and the equations of motion.

Introducting Eqs. (125) into Eqs. (44), (46), and (47), multiplying Eqs. (44) and the three components of Eq. (47) by Q_{Rt}, $Q_{\phi t}$, Q_{zt} and P_t, and integrating over the volume of liquid, we obtained an algebraic eigenvalue problem of order 4n+6 in the coefficients A_s, B_s, C_s, D_s, the angles of $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$, with $\hat{\lambda}$ as a parameter. The eigenvalue problem and the angular velocities $\hat{\omega}_1 = \hat{\lambda}\omega_N\hat{\theta}_1$, $\hat{\omega}_2 = \hat{\lambda}\omega_N\hat{\theta}_2$, and $\hat{\omega}_3 = \hat{\lambda}\omega_N\hat{\theta}_3$ can be written in the form

$$\hat{\lambda}\, E\underset{\sim}{x} + F\underset{\sim}{x} = \underset{\sim}{0} \tag{126}$$

A solution of the eigenvalue problem (126) was attempted by the QR method but, in spite of considerable time and effort, the results proved unsatisfactory. In particular, the obtained eigenvalues $\hat{\lambda}$ have real parts, which is in contradiction with the expectation of a purely oscillatory motion for an inviscid stable problem. The conclusion that can be drawn from this is that a Galerkin approach is not feasible for this problem because of the difficulty in selecting a satisfactory set of admissible functions.

Because the semi-analytical approach described above did not lead to satisfactory results, a completely numerical solution of the inviscid problem with the modified boundary condittions may be attempted by means of finite differences or finite elements. Although that approach may permit

a ready extension to other tank geometries, the work promises to be extremely tedious, because a three-dimensional finite-difference grid or a three-dimensional finite element must be used. We note, in passing, that truly three-dimensional numerical solutions of fluid dynamical problems are scarce.

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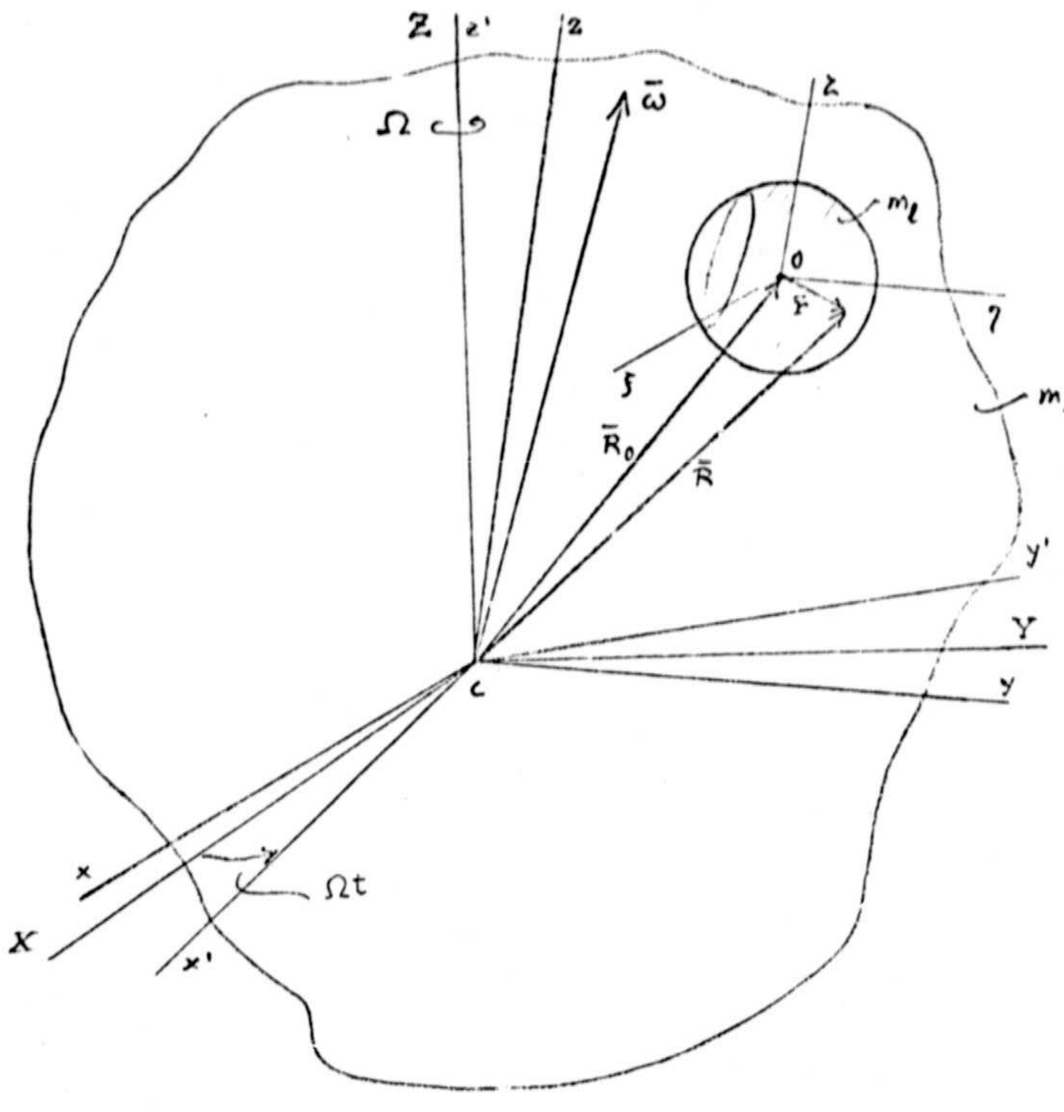

Figure 1